: $(x^4 e^x - 2\,mxy^2)\,dx + 2mx^2y\,dy = 0$

(Q22)

Here,

$$M = x^4 e^x - 2\,mxy^2$$

$$N = 2mx^2y$$

$$\frac{\partial M}{\partial y} = -4\,mxy\,; \qquad \frac{\partial N}{\partial x} = 4\,mxy$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{-8\,mxy}{2\,mx^2y} = -\frac{4}{x} = \text{function of x alone}$$

$$\text{I.F.} = e^{-4\int \frac{1}{x}dx} = e^{-4\log x} = x^{-4} = \frac{1}{x^4}$$

$$\frac{x^4 e^x - 2\,mxy^2}{x^4}\,dx + \frac{2\,mx^2y}{x^4}\,dy = 0$$ is exact

∴ General solution is

$$\int e^x\,dx - 2\,my^2 \int \frac{1}{x^3}\,dx = c$$

$$\therefore \quad e^x + \frac{my^2}{x^2} = c$$

(23) Solve : $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$

Sol. :

Here $M = y^4 + 2y,$

$N = xy^3 + 2y^4 - 4x$

$$\therefore \frac{\partial M}{\partial y} = 4y^3 + 2\ ,\ \frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y}$$

$$\therefore \text{I.F.} = e^{-3\int \frac{1}{y}dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$$

$$\therefore \frac{(y^4+2y)}{y^3}dx + \frac{(xy^3+2y^4-4x)}{y^3}dy = 0 \text{ is exact}$$

$\therefore$ General solution is

$$\left(y + \frac{2}{y^2}\right)\int dx + 2\int y\,dy = c$$

$$\left(y + \frac{2}{y^2}\right)x + y^2 = c$$

(24) Solve : $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)\,dy = 0.$

:

$$\text{Here} \quad M = \frac{y}{x}\sec y - \tan y$$

$$N = \sec y \log x - x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \cdot \tan y - \sec^2 y$$

$$\frac{\partial N}{\partial x} = \frac{\sec y}{x} - 1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -1 - \frac{y}{x}\sec y \tan y + \sec^2 y = \tan y\left(\tan y - \frac{y}{x}\sec y\right)$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\tan y\left(\tan y - \frac{y}{x}\sec y\right)}{\frac{y}{x}\sec y - \tan y} = -\tan y = \text{function of } y \text{ only}$$

$$\text{I.F.} = e^{-\int \tan y \cdot dy} = e^{-\log \sec y} = \cos y$$

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)dx + \cos y\,(\sec y \log x - x)\,dx = 0 \quad \text{is exact}$$

General solution is

$$y\int \frac{1}{x}\,dx - \sin y \int dx = c$$

$$y \log x - x \sin y = c$$

11 : If y^n is an I.F. of the equation.
2/95 $y(2x^2y + e^x)\,dx - (e^x + y^2)\,dy = 0$. Find n and hence solve the equation.

$\because$ y^n is I.F.

$\therefore$ equation will be exact after multiplying by it

i.e. $y^{n+1}(2x^2y + e^x)\,dx - y^n(e^x + y^2)\,dy = 0$ is exact

Here, $M = 2x^2y^{n+2} + y^{n+1} \cdot e^x$

and $N = -(e^x y^n + y^{n+2})$

$\therefore$ $\dfrac{\partial M}{\partial y} = 2(n+2)x^2y^{n+1} + (n+1)y^n e^x$

$$\frac{\partial N}{\partial x} = -e^x y^n$$

and $\because$ $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore 2(n+2)x^2 y^{n+1} + (n+1)y^n e^x = -e^x y^n$$

$$\therefore \quad (n+2)[2x^2 y^{n+1} + y^n e^x] = 0$$

$$\therefore \quad n = -2$$

($\because$ $2x^2 y^{n+1} + y^n e^x \neq 0$. If it is zero, then it will form a solut
can be verified that it is not a solution of the given equation.)

$$\therefore \quad \frac{1}{y^2} \text{ is I.F.}$$

Multiplying the given equation by $\frac{1}{y^2}$, we get

$$\frac{y}{y^2}(2x^2 y + e^x)\,dx - \frac{(e^x + y^2)}{y^2}\,dy = 0 \text{ is exact,}$$

Its General solution is

$$2\int x^2\,dx + \frac{1}{y}\int e^x\,dx - \int dy = c$$

i.e. $$\frac{2}{3}x^3 + \frac{e^x}{y} - y = c$$ is General solution

12 : If $(x+y+1)^n$ is I.F. for the equation,
26) $(2xy - y^2 - y)\,dx + (2xy - x^2 - x)\,dy = 0$. Find n and solve the equation.

. : Multiplying the equation by $(x+y+1)^n$, we get,

$(x+y+1)^n(2xy - y^2 - y)\,dx + (x+y+1)^n(2xy - x^2 - x)\,dy = 0$ is exact equation

Here,

$$M = (x+y+1)^n(2xy - y^2 - y)$$

$$N = (x+y+1)^n(2xy - x^2 - x)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \quad n(x+y+1)^{n-1}(2xy - y^2 - y) + (x+y+1)^n(2x - 2y - 1)$$

$$= n(x+y+1)^{n-1}(2xy - x^2 - x) + (x+y+1)^n(2y - 2x - 1)$$

Cancelling $(x+y+1)^{n-1}$ throughout, we get

$$n(2xy - y^2 - y) + (x+y+1)(2x - 2y - 1) = n(2xy - x^2 - x) + (x+y+1)(2y - 2x - 1)$$

$$n[x^2 + x - y^2 - y] + 4[x^2 + x - y^2 - y] = 0$$

$\therefore \quad (n + 4)\,(x^2 + x - y^2 - y) = 0$

$\therefore \quad n + 4 = 0 \qquad (\because x^2 + x - y^2 - y \neq 0)$

$\therefore \quad n = -4$

$\therefore \quad \text{I.F.} = (x + y - 1)^{-4}$

Premultiplying the equation by I.F., we get

$(x + y + 1)^{-4}\,(2xy - y^2 - y)\,dx + (x + y + 1)^{-4}\,(2xy - x^2\,x)\,dy = 0$ is exact

$\therefore$ General solution is,

$$2y \int \frac{x}{(x + y + 1)^4}\,dx - y\,(y + 1) \int (x + y + 1)^{-4} \cdot dx = c$$

$$\text{i.e. } 2y \int \frac{(x + y + 1) - (y + 1)}{(x + y + 1)^4}\,dx - y\,(y + 1)\,\frac{(x + y + 1)^{-3}}{-3} = c$$

$$\text{i.e. } 2y \left[\int (x + y + 1)^{-3} \cdot dx - (y + 1) \int (x + y + 1)^{-4} \cdot dx \right] + \frac{y\,(y + 1)}{3} \cdot \frac{1}{(x + y + 1)^3} = c$$

$$\text{i.e. } 2y \left[\frac{(x + y + 1)^{-2}}{-2} - (y + 1)\,\frac{(x + y + 1)^{-3}}{-3} \right] + \frac{y\,(y + 1)}{3\,(x + y + 1)^3} = c$$

Simplifying we get,

$$\frac{xy}{(x + y + 1)^3} = c \quad \text{is General solution}$$

18 : If the equation Mdx + Ndy = 0 can be made exact by means of I.F. a function of x alone then show that $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is independent of y.

. : Let f (x) which is a function of x only be the I.F. of the equation Mdx + Ndy = 0

$\therefore \quad f(x)\,M\,dx + f(x)\,N\,dy = 0$ is exact

$$\therefore \quad \frac{\partial}{\partial y}[f(x)M] = \frac{\partial}{\partial x}[f(x)N]$$

$$\therefore \quad f(x)\frac{\partial M}{\partial y} = f'(x)N + f(x)\frac{\partial N}{\partial x}$$

$$\therefore \quad f(x)\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right] = f'(x)N$$

$$\therefore \quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{f'(x)}{f(x)}$$

= a function of x alone is independent of y [redacted] : If μ is an I.F. of the equation $Mdx + Ndy = 0$ then show that

(28)

$$\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = N\left(\frac{\partial \mu}{\partial x}\right)-M\left(\frac{\partial \mu}{\partial y}\right)$$

l. : The equation,

$$\mu M\,dx + \mu N\,dy = 0 \qquad \text{is exact}$$

$$\therefore \quad \frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\therefore \quad \mu\frac{\partial M}{\partial y}+\frac{\partial \mu}{\partial y}M = \frac{\partial \mu}{\partial x}N+\mu\frac{\partial N}{\partial x}$$

$$\therefore \quad \mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}$$

$(2x \log x - xy)\, dy + 2y \cdot dx = 0$

:

$$M = 2y, \qquad N = 2x \log x - xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2; \qquad \frac{\partial N}{\partial x} = 2 \log x + 2 - y$$

$$\therefore \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - (2 \log x + 2 - y)}{2x \log x - xy} = \frac{-(2 \log x - y)}{x(2 \log x - y)}$$

$$= -\frac{1}{x} \text{ is a function of } x$$

$$\therefore \quad \text{I. F.} = e^{-\int \frac{1}{x} \cdot dx}$$

$$= e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$\therefore \quad \frac{1}{x}(2x \log x - xy)\, dy + \frac{2y}{x} dx = 0 \text{ is exact.}$$

Hence General Solution is,

$$2y \int \frac{1}{x} dx + \int (-y)\, dy = c$$

$$\therefore \quad 2y \log x - \frac{y^2}{2} = c$$

is General Solution.

(31) $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0.$ **(May 96, 4 Marks)**

: We have,

$$M = 2xy^4e^y + 2xy^3 + y$$

$$N = x^2y^4e^y - x^2y^2 - 3x$$

$$\therefore \frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\left[2xy^4e^y - 2xy^2 - 3\right] - \left[8xy^3e^y + 2xy^4e^y + 6xy^2 + 1\right]}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{y\left(2xy^3e^y + 2xy^2 + 1\right)}$$

$$= \frac{-4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = -\frac{4}{y} = \text{function of } y$$

$$\therefore \text{I. F.} = e^{-4\int \frac{1}{y}dy} = e^{-4\log y}$$

$$= e^{\log(y^{-4})} = \frac{1}{y^4}$$

Multiplying the equation by $\frac{1}{y^4}$; we get

$$\frac{1}{y^4}(2xy^4e^y + 2xy^3 + y)\,dx + \frac{1}{y^4}(x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$$

is exact

Hence General solution is

$$2e^y\int x\,dx + \frac{2}{y}\int x\,dx + \frac{1}{y^3}\int dx = c$$

$$\therefore \quad x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c \text{ is General Solution.}$$

Ex. 12 : $y(xy\sin xy + \cos xy)\,dx + x(xy\sin xy - \cos xy)\,dy = 0.$

Sol. : Here

$$M = y(xy\sin xy + \cos xy)$$

$$N = x(xy\sin xy - \cos xy)$$

$$\therefore \quad I.F. = \frac{1}{Mx - Ny} = \frac{1}{xy[2\cos xy]}$$

$$= \frac{1}{2xy\cos xy}$$

Multiplying by I. F., we get

$$\frac{y(xy\sin xy + \cos xy)}{2xy\cos xy}dx + \frac{x(xy\sin xy - \cos xy)}{2xy\cos xy}dy = 0$$

On simplification,

$$\frac{y}{2}\int \tan xy \cdot dx + \frac{1}{2}\int\frac{1}{x}dx - \frac{1}{2}\int\frac{1}{y}dy = c$$

$$\text{i.e. } \frac{y}{2}\frac{\log(\sec xy)}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\therefore \quad \log\left[\frac{x}{y}\sec xy\right] = 2c$$

$$\therefore \quad \frac{x}{y}\sec(xy) = e^{2c} = c_1$$ is G

Ex. 14 : $y\,dx + x(1 - 3x^2y^2)\,dy = 0.$ (Dec. 9

Sol. : Here,

$$M = y, \qquad N = x(1 - 3x^2y^2)$$

$$\therefore \quad I.F. = \frac{1}{Mx - Ny} = \frac{1}{xy\left[1 - 1 + 3x^2y^2\right]}$$

$$= \frac{1}{3x^3y^3}$$

Multiplying the equation by I.F.,

$$\frac{y}{3x^3y^3}\,dx + \frac{1}{3x^3y^3}\,x(1 - 3x^2y^2)\,dy = 0 \qquad \text{is exact.}$$

∴ General Solution is,

$$\frac{1}{3y^2}\int x^{-3}\cdot dx - \int \frac{dy}{y} = c$$

$$\text{i.e.,} \quad -\frac{1}{6x^2y^2} - \log y = c$$

$$\text{i.e.} \quad \frac{1}{6x^2y^2} + \log y = -c = c_1 \text{ (say)}$$

Ex. 34: If $(x+y)^n$ is an integrating factor of the equation, $(4x^2+2xy+6y)\,dx+(2x^2+9y+3x)\,dy=0$. Find this I.F. and the solution of the equation. **(Dec. 97, 6 Marks)**

Sol. : We multiply the equation by I. F. $(x+y)^n$, and get

$$(x+y)^n(4x^2+2xy+6y)\,dx+(x+y)^n(2x^2+9y+3x)\,dy=0$$

Here, $M = (x+y)^n(4x^2+2xy+6y)$

$N = (x+y)^n(2x^2+9y+3x)$

$$\therefore \quad \frac{\partial M}{\partial y} = n(x+y)^{n-1}(4x^2+2xy+6y)+(x+y)^n(2x+6)$$

and $$\frac{\partial N}{\partial x} = n(x+y)^{n-1}(2x^2+9y+3x)+(x+y)^n(4x+3)$$

Since the factor $(x+y)^n$ makes the equation exact, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore n(x+y)^{n-1}(4x^2+2xy+6y)+(x+y)^n(2x+6) = n(x+y)^{n-1}(2x^2+9y+3x)+(x+y)^n(4x+3)$$

$$\therefore n(x+y)^{n-1}\left[4x^2+2xy+6y-2x^2-9y-3x\right] + (x+y)^n[2x+6-4x-3] = 0$$

$$\therefore n(x+y)^{n-1}\left[2x^2+2xy-3y-3x\right]+(x+y)^n[-2x+3]=0$$

$$\therefore n(x+y)^{n-1}[(2x-3)(x+y)]-(x+y)^n(2x-3)=0$$

$$\therefore (n-1)(x+y)^n(2x-3)=0$$

since $(x+y)\neq 0,\ (2x-3)\neq 0$

$$\therefore \quad n=1$$

$$\therefore \quad \text{I. F.} = (x+y)$$

Multiplying the equation by $(x+y)$; we get

$$\therefore \quad (x+y)(4x^2+2xy+6y)\,dx+(x+y)(2x^2+9y+3x)\,dy=0 \text{ is exact.}$$

Hence, its General Solution is given by

$$4\int x^3\cdot dx+6y\int x^2\cdot dx+6y\int x\cdot dx+2y^2\int x\cdot dx+6y^2\int dx+9\int y^2\,dy=c$$

i.e., $x^4+2y\,x^3+3yx^2+x^2y^2+6xy^2+3y^3=c$ is General Solution

Ex. 33 : If x^n is an integrating factor of the equation :

$(x^4 e^x - 2mxy^2)\,dx + (2mx^2 y)\,dy = 0$ find n and solve the eq

Sol. : Since x^n is I. F. of the equation, we multiply the equati

$x^n(x^4 e^x - 2mxy^2)\,dx + x^n(2mx^2 y)\,dy = 0$

$$\therefore \quad M = x^{n+4} \cdot e^x - 2mx^{n+1} \cdot y^2$$

$$\text{and} \quad N = 2mx^{n+2} \cdot y$$

$$\text{Now,} \quad \frac{\partial M}{\partial y} = -4mx^{n+1} \cdot y$$

$$\text{and} \quad \frac{\partial N}{\partial x} = 2m(n+2)x^{n+1} \cdot y$$

Since the equation is exact,

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \quad -4mx^{n+1} \cdot y = 2m(n+2) \cdot x^{n+1} \cdot y$$

$$\therefore \quad n+2 = -2$$

$$\therefore \quad n = -4$$

$$\therefore \quad x^{-4} = \frac{1}{x^4} \text{ is I.F. of the equation}$$

We multiply the equation by $\frac{1}{x^4}$,

$$\frac{1}{x^4}(x^4 e^x - 2m\,xy^2)\,dx + \frac{1}{x^4}(2\,mx^2 y)\,dy = 0 \text{ is exact.}$$

∴ General Solution is,

$$\int e^x \cdot dx - 2\,my^2 \int \frac{1}{x^3}\,dx = c$$

$$\therefore \quad e^x - 2\,my^2\,\frac{x^{-2}}{-2} = c$$

$$\therefore \quad e^x + \frac{my^2}{x^2} = c$$

Ex. 27 : $\frac{dy}{dx} = -\frac{(x^2 y^3 + 2y)}{(2x - 2x^3 y^2)}$

Sol. : The equation can be written as,

$$(x^2 y^3 + 2y)\,dx + (2x - 2x^3 y^2)\,dy = 0$$

$$\text{i.e., } x^2 y^2 (y\,dx - 2x\,dy) + 2(y\,dx + x\,dy) = 0$$

Dividing throughout by $x^3 y^3$, we get

$$\left(\frac{y\,dx - 2x\,dy}{xy}\right) + 2\frac{d(xy)}{x^3 y^3} = 0$$

$$\therefore \quad \frac{dx}{x} - 2\frac{dy}{y} + 2\frac{d(xy)}{(xy)^3} = 0$$

The equation is in variable separable form, integrating,

$$\log x - 2\log y + 2\frac{(-1)}{2x^2 y^2} = c$$

$$\therefore \quad \log\left(\frac{x}{y^2}\right) - \frac{1}{x^2 y^2} = c \quad \text{is General Solution}$$

Ex. : If $f(x)$ a function of x only is an integrating factor of

$$(x^4 e^x - 2\,m\,xy^2)\,dx + 2\,m\,x^2 y\,dy = 0$$

(M.U. 19994)

find $f(x)$ and then solve the equation.

Sol. : Multiplying the equation by $f(x)$,

$$f(x) \cdot (x^4 e^x - 2\,m\,xy^2)\,dx + f(x) \cdot 2\,mx^2 y\,dy = 0$$

But this is exact. $\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\therefore \quad f(x)(-4\,m\,xy) = f(x)\,4m\,xy + f'(x)\,2m\,x^2 y$$

$$\therefore \quad -4m\,xy \cdot f(x) = f'(x) \cdot m\,x^2 y$$

$$\therefore \quad f'(x) = -\frac{4}{x} f(x) \qquad \therefore \quad \frac{f'(x)}{f(x)} = -\frac{4}{x}$$

$$\therefore \quad \int \frac{f'(x)}{f(x)}\,dx = -4 \int \frac{dx}{x}$$

$$\therefore \quad \log f(x) = -4 \log x = \log x^{-4} \qquad \therefore \quad f(x) = x^{-4}$$

Now, multiply the given equation by x^{-4} and you get (B) of the above example. Now proceed as above.

Ex42 : Solve : $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

Sol. : The equation can be written as

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

is linear, in y, with $P = \frac{1}{\sqrt{x}}$

$$\therefore \quad \text{I.F.} = e^{\int P \cdot dx} = e^{\int \frac{1}{\sqrt{x}} \cdot dx} = e^{2\sqrt{x}} \qquad \text{... (1)}$$

$\therefore$ General solution is

$$y\, e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot dx + c$$

$$= \int \frac{1}{\sqrt{x}}\, dx + c = 2\sqrt{x} + c \quad \text{is general solution}$$

Ex43 : Solve : $\cosh x \cdot \frac{dy}{dx} = 2\cosh^2 x \cdot \sinh x - y \sinh x$

Sol. : The equation can be written as

$$\frac{dy}{dx} + \frac{\sinh x}{\cosh x}\, y = \frac{2\cosh^2 x \cdot \sinh x}{\cosh x}$$

i.e. $\frac{dy}{dx} + \tanh x \cdot y = 2 \sinh x \,.\, \cosh x$ is linear in y with P =

$\therefore$ I.F. $= e^{\int P \cdot dx} = e^{\int \tanh x \cdot dx} = e^{\log \cosh x} = \cosh x$

$\therefore$ General solution is

$$y (\cosh x) = \int 2 \sinh x \cdot \cosh^2 x \cdot dx + c$$

$$= \frac{2}{3} \cosh^3 x + c$$

i.e. $3y \cosh x = 2 \cosh^3 x + 3c$ is general solution

Ex. 45: Solve : $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\dfrac{dy}{dx}=0$

Sol. :

Note : Presence of x and no other term in x indicates that equation may be linear variable x.

The equation can be written as,

$$\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx} = -(1+y^2)$$

$$\therefore \quad x-e^{\tan^{-1}y} = -(1+y^2)\frac{dx}{dy}$$

$$\therefore \quad \frac{dx}{dy}+\frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$ is linear independent variable

Here $$P = \frac{1}{1+y^2}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\therefore$ General solution is,

$$x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c$$

i.e. $$x\,e^{\tan^{-1}y} = \int e^{\tan^{-1}y}\frac{e^{\tan^{-1}y}}{1+y^2}dy + c$$

$$= \int \frac{e^{2\tan^{-1}y}}{1+y^2}dy + c$$

Let $$\tan^{-1}y = t$$

$$\therefore \quad \frac{1}{1+y^2}dy = dt$$

Ex. 16 Solve: $\frac{dy}{dx} - y\tan x = y^4 \sec x$

Sol. : Dividing by y^4, we get

$$y^{-4}\frac{dy}{dx} - y^{-3}\tan x = \sec x$$

$$\text{Let} \quad y^{-3} = u \quad \therefore \quad -3y^{-4}\frac{dy}{dx} = \frac{du}{dx}$$

∴ (1) becomes,

$$-\frac{1}{3}\frac{du}{dx} - u\tan x = \sec x$$

i.e. $$\frac{du}{dx} + 3u\tan x = -3\sec x$$

is linear independent variable u.

∴ $$\text{I.F.} = e^{3\int \tan x \cdot dx} = e^{3\log\sec x} = \sec^3 x$$

∴ General solution is,

$$u(\sec^3 x) = \int \sec^3 x \cdot (-3\sec x)\cdot dx + c$$

$$= -3\int (1+\tan^2 x)\cdot \sec^2 x \cdot dx + c$$

Let $\tan x = t$ ∴ $\sec^2 x \cdot dx = dt$

∴ $$u\cdot\sec^3 x = -3\int (1+t^2)\cdot dt + c$$

∴ $$u\sec^3 x = -3t - t^3 + c$$

i.e. $$y^{-3}\cdot\sec^3 x + 3\tan x + \tan^3 x = c \qquad (\because u = y^{-3})$$

or $$\frac{\sec^3 x}{y^3} + 3\tan x + \tan^3 x = c$$ is General solution

Ex.17 :

Solve : $\frac{dy}{dx} - xy = y^2 e^{\frac{-x^2}{2}} \cdot \log x$

Sol. : The equation is of Bernoulli's form. Dividing by y^2 , we get

$$y^{-2}\frac{dy}{dx} - x y^{-1} = e^{\frac{-x^2}{2}} \cdot \log x$$

Let $y^{-1} = u \quad \therefore \quad -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$

$\therefore$ The equation becomes,

$$-\frac{du}{dx} - xu = e^{\frac{-x^2}{2}} \cdot \log x$$

i.e. $$\frac{du}{dx} + xu = -e^{\frac{-x^2}{2}} \cdot \log x$$

is linear in u

$\therefore$ $$\text{I.F.} = e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

∴ General solution is,

$$u \cdot e^{\frac{x^2}{2}} = -\int e^{\frac{x^2}{2}} \cdot e^{\frac{-x^2}{2}} \cdot \log x \cdot dx + c$$

$$= -\int \log x \cdot dx + c = -(x \log x - x) + c$$

$$u = y^{-1} = \frac{1}{y}$$

General solution is,

$$\frac{1}{y} e^{\frac{x^2}{2}} + x \log x - x = c$$

Ex 13 : Solve: $xy\,(1 + xy^2)\dfrac{dy}{dx} = 1$

Sol. : The equation can be written as

$$\frac{dx}{dy} = xy + x^2 y^3$$

i.e. $$\frac{dx}{dy} - xy = x^2 y^3$$

is Bernoulli's equation in x.

Dividing by x^2 throughout, we get

$$x^{-2}\frac{dx}{dy} - x^{-1}\cdot y = y^3$$

Let $x^{-1} = u \quad \therefore \quad -x^{-2}\dfrac{dx}{dy} = \dfrac{du}{dy}$

$\therefore$ The equation becomes,

$$-\frac{du}{dy} - u\,y = y^3$$

$\therefore$ $\dfrac{du}{dy} + uy = -y^3$ is linear independent variable u,

$\therefore$ $$\text{I.F.} = e^{\int y\,dy} = e^{\frac{y^2}{2}}$$

$\therefore$ General solution is,

$$u\,e^{\frac{y^2}{2}} = -\int y^3 \cdot e^{\frac{y^2}{2}} \cdot dy + c$$

Let $\dfrac{y^2}{2} = t \quad \therefore \quad y\,dy = dt$

$\therefore$ (1) becomes,

$$u\,e^{\frac{y^2}{2}} = -\int 2t \cdot e^t \cdot dt + c = -2\,[\,t\,e^t - e^t\,] + c$$

$$\frac{1}{x} e^{\frac{y^2}{2}} + 2\left[\frac{y^2}{2} e^{\frac{y^2}{2}} - e^{\frac{y^2}{2}}\right] = c$$

$$\frac{1}{x} e^{\frac{y^2}{2}} + y^2 e^{\frac{y^2}{2}} - 2 e^{\frac{y^2}{2}} = c$$ is General solution

Ex. 19 : Solve : $\sin y \cdot \frac{dy}{dx} = (1 - x\cos y) \cdot \cos y$

Sol. : The equation can be written as

$$\sin y \cdot \frac{dy}{dx} = \cos y - x\cos^2 y$$

Dividing throughout by $\cos^2 y$, we get

$$\frac{\sin y}{\cos^2 y}\frac{dy}{dx} = \frac{\cos y}{\cos^2 y} - x$$

i.e. $\sec y \cdot \tan y \dfrac{dy}{dx} - \sec y = -x$

[is of the form, $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ where $f(y) = \sec y$, $P = -1$, $Q = -x$

Let $\sec y = u$

$$\sec y \cdot \tan y \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$\therefore$ (1) becomes,

$$\frac{du}{dx} - u = -x$$

$\therefore$ $\text{I.F.} = e^{-\int dx} = e^{-x}$

$\therefore$ General solution is,

$$u\,e^{-x} = -\int x\,e^{-x} \cdot dx + c$$

i.e. $u\,e^{-x} = -[\,x(-e^{-x}) - (1)(e^{-x})\,] + c$

(integra

$\because$ $u = \sec y$

General solution is $\sec y = x + 1 + c\,e^{x}$

Ex. 2 : Solve : $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \cdot \sin y$

Sol. : Dividing the equation by tan y · sin y , we get

$$\frac{1}{\tan y \cdot \sin y}\frac{dy}{dx} + \frac{1}{x}\frac{\tan y}{\tan y \cdot \sin y} = \frac{1}{x^2}$$

i.e. $$\cot y \cdot \text{cosec } y \cdot \frac{dy}{dx} + \frac{1}{x} \cdot \text{cosec } y = \frac{1}{x^2}$$

Let $$\text{cosec } y = u$$

$$\therefore \quad -\text{cosec } y \cdot \cot y \cdot \frac{dy}{dx} = \frac{du}{dx}$$

∴ Equation becomes

$$-\frac{du}{dx} + \frac{1}{x} \cdot u = \frac{1}{x^2}$$

i.e. $$\frac{du}{dx} - \frac{1}{x} \cdot u = -\frac{1}{x^2}$$

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

. **General solution is,**

$$u \cdot \frac{1}{x} = -\int \frac{1}{x^2} \cdot \frac{1}{x} \cdot dx + c = \frac{1}{2x^2} + c$$

$$\frac{1}{x \sin y} = \frac{1}{2x^2} + c$$ is General solution.

Ex. : $\frac{dy}{dx} - x^3 \cos^2 y = -x \sin 2y$

Sol. : The equation can be written as

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\therefore \quad \frac{1}{\cos^2 y}\frac{dy}{dx} + x\left(\frac{2 \sin y \cos y}{\cos^2 y}\right) = x^3$$

i.e. $$\sec^2 y \cdot \frac{dy}{dx} + 2 \tan y \cdot x = x^3$$

Let $$\tan y = u$$

$$\therefore \quad \sec^2 y \cdot \frac{dy}{dx} = \frac{du}{dx}$$ and hence the equation becomes

$$\frac{du}{dx} + 2ux = x^3$$ is linear in u

$$\text{I.F.} = e^{2\int x \cdot dx} = e^{x^2}$$

$\therefore$ G.S. is $$u \cdot e^{x^2} = \int x^3 \cdot e^{x^2} \cdot dx + c$$

Let $x^2 = t \quad \therefore \; 2x\,dx = dt$

$$= \int t \cdot e^t \cdot \frac{dt}{2} + c = \frac{1}{2}[te^t - e^t] + c$$

$\therefore$ General solution is,

$$\tan y \cdot e^{x^2} = \frac{1}{2}(x^2 - 1)e^{x^2} + c$$

or $$2 \tan y = x^2 - 1 + c_1 e^{-x^2} \qquad (2c = c_1)$$

Ex. 5): $\frac{dy}{dx} + \frac{y}{x} \cdot \log y = \frac{y}{x^2} (\log y)^2$

Sol. : Dividing by $y(\log y)^2$, we get

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{\log y} = \frac{1}{x^2}$$

Let $$\frac{1}{\log y} = u$$

$$\therefore \quad -\frac{1}{y(\log y)^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \quad -\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2}$$

i.e. $$\frac{du}{dx} - \frac{u}{x} = -\frac{1}{x^2}$$ is linear in v

$$\therefore \quad \text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

$\therefore$ General solution is

$$u\left(\frac{1}{x}\right) = -\int \frac{1}{x} \cdot \frac{1}{x^2} dx + c = \frac{1}{2x^2} + c$$

i.e. $$\frac{1}{x \log y} = \frac{1}{2x^2} + c$$ is G.S.

Ex. 5 : Solve : $\frac{d}{dx}\left(\frac{dv}{dx}+\frac{2v}{x}\right)=0$

Sol. :

Let $w = \frac{dv}{dx}+\frac{2v}{x}$

$\frac{dw}{dx} = 0$ integrating

$w = c_1$, c_1 is constant

is $\frac{dv}{dx}+\frac{2v}{x} = c_1$ is linear in v

$\text{I.F.} = e^{2\int \frac{1}{x}dx} = e^{2\log x} = x^2$

General solution is

$$V x^2 = c_1 \int x^2\, dx + c_2$$

$$V x^2 = c_1 \frac{x^3}{3} + c_2$$ is Gene

10: $2\cos x \cdot \frac{dy}{dx} + 4y\sin x = \sin 2x$ given that $y = 0$ when $x = \frac{\pi}{3}$ VI

.: The equation is

$$\frac{dy}{dx} + \frac{2\sin x}{\cos x}\cdot y = \frac{2\sin x\cdot\cos x}{2\cos x} = \sin x$$

i.e. $\frac{dy}{dx} + 2\tan x\cdot y = \sin x$ is linear in y

$$\text{I.F.} = e^{2\int \tan x\cdot dx} = e^{2\log\sec x} = \sec^2 x$$

∴ General solution is

$$y\sec^2 x = \int \sin x\cdot\sec^2 x\cdot dx + c = \int \sec x\cdot\tan x\cdot dx + c$$

$$= \sec x + c$$

∴ $y = \cos x + c\cos^2 x$ is General solution ... (1)

Now, when $x = \frac{\pi}{3}$, $y = 0$; substitute in (1)

∴ $0 = \frac{1}{2} + c\,\frac{1}{4}$

∴ $c = -2$

∴ $y = \cos x - 2\cos^2 x$ is Particular Solution

.35 $dr + (2r\cot\theta + \sin 2\theta)\cdot d\theta = 0.$ VI

l. : We have,

$$\frac{dr}{d\theta} + 2r\cot\theta + \sin 2\theta = 0$$

$$\therefore \quad \frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta \qquad \text{is linear in r.}$$

$$\therefore \quad \text{I.F.} = e^{\int 2\cot\theta\cdot d\theta} = e^{2\log\sin\theta} = \sin^2\theta$$

General Solution is given by,

$$r(\sin^2\theta) = -\int \sin 2\theta\cdot\sin^2\theta\cdot d\theta + c$$

$$= -2\int \sin^3\theta\cdot\cos\theta\cdot d\theta + c$$

$$= -2\left(\frac{\sin^4\theta}{4}\right) + c$$

$$\text{Using} \quad \int f^n f' = \frac{f^{n+1}}{n+1} = -\frac{\sin^4\theta}{2} + c$$

General Solution is given by,

$$2r\sin^2\theta + \sin^4\theta = 2c = c_1$$

56 $\frac{dy}{dx} = (e^{x-y})(e^x - e^y)$.

I. : The given equation is,

$$\frac{dy}{dx} = e^x \cdot e^{-y}(e^x - e^y)$$

$$\therefore \quad e^y \cdot \frac{dy}{dx} = e^x(e^x - e^y)$$

$$\text{i.e.;} \quad e^y \frac{dy}{dx} = e^{2x} - e^x \cdot e^y$$

$$\therefore \quad e^y \frac{dy}{dx} + e^x \cdot e^y = e^{2x}$$

Let $e^y = u$,

$$\therefore \quad e^y \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \quad \frac{du}{dx} + ue^x = e^{2x}$$

$$\therefore \quad \text{I. F.} = e^{\int e^x \cdot dx} = e^{e^x}$$

$\therefore$ General Solution is,

$$u\,e^{e^x} = \int e^{2x} \cdot e^{e^x} \cdot dx + c$$

where c is arbitrary constant.

Let $e^x = t$

$$\therefore \quad e^x\,dx = dt$$

$\therefore$ General Solution is,

$$u\,e^{e^x} = \int t\,e^t \cdot dt + c$$

Integrating by parts,

$$= \left[\, t\,e^t - e^t \,\right] + c$$

$\therefore$ General Solution is,

$$u\,e^{e^x} = e^x \cdot e^{e^x} - e^{e^x} + c$$

$$\text{i.e.,} \quad e^y = e^x - 1 + ce^{-e^x}$$

$$\frac{dx}{dy} + \frac{x(\sec y + \tan y)}{(1 + \sin y)} = \frac{2y \cos y}{1 + \sin y}$$

i.e., $$\frac{dx}{dy} + \frac{x}{(1 + \sin y)}\left(\frac{1 + \sin y}{\cos y}\right) = \frac{2y \cos y}{1 + \sin y}$$

i.e., $$\frac{dx}{dy} + x \sec y = \frac{2y \cos y}{1 + \sin y}$$

The equation is linear in x.

$$\therefore \quad \text{I. F.} = e^{\int \sec y \cdot dy} = e^{\log(\sec y + \tan y)}$$

$$= (\sec y + \tan y) = \frac{(1 + \sin y)}{\cos y}$$

Hence, the General Solution is given by,

$$x \cdot \frac{(1 + \sin y)}{\cos y} = \int \frac{(1 + \sin y)}{\cos y} \cdot \frac{2y \cos y}{(1 + \sin y)} dy$$

$$= y^2 + c$$

or $$x(1 + \sin y) = y^2 \cos y + c \cdot \cos y$$

10: $(1-x^2)\frac{dy}{dx}+2xy = x\sqrt{1-x^2}$.

. : The equation is

$$\frac{dy}{dx}+\frac{2xy}{1-x^2} = \frac{x\sqrt{1-x^2}}{(1-x^2)} = \frac{x}{\sqrt{1-x^2}}$$

$$\text{I. F.} = e^{\int \frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{-1}} = \frac{1}{1-x^2}$$

$$y\frac{1}{(1-x^2)} = \int \frac{1}{(1-x^2)}\frac{x}{\sqrt{1-x^2}}dx + c$$

$$= -\frac{1}{2}\int(-2x)(1-x^2)^{-3/2}\cdot dx + c$$

$$= -\frac{1}{2}\frac{(1-x^2)^{-1/2}}{(-1/2)} + c = \frac{1}{\sqrt{1-x^2}} + c$$

$$\therefore \quad y = \sqrt{1-x^2} + c(1-x^2) \text{ is General Solution.}$$

Q: $\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0.$

Sol.: The equation is,

$$\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$$

$$\text{Let } u = \tan^{-1}y$$

$$\therefore \frac{du}{dx} = \frac{1}{(1+y^2)}\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = (1+y^2)\frac{du}{dx}$$

The equation,

$$(1+y^2)\frac{du}{dx} + (2xu - x^3)(1+y^2) = 0$$

$$\therefore \frac{du}{dx} + 2xu = x^3 \text{ is linear in u.}$$

$$\therefore \text{I. F.} = e^{\int 2x\,dx} = e^{x^2}$$

General Solution is,

$$u\,e^{x^2} = \int e^{x^2}\cdot x^3\cdot dx + c$$

$$\text{Let } x^2 = t \qquad \therefore 2xdx = dt$$

$$= \int e^t \cdot t \cdot \frac{dt}{2} + c$$

$$\therefore \quad u\, e^{x^2} = \frac{1}{2}\left[\, t\, e^{t} - e^{t} \,\right] + c$$

∴ General Solution is

$$\tan^{-1} y \cdot e^{x^2} = \frac{1}{2}\left[\, x^2 - 1 \,\right] e^{x^2} + c$$

$$\text{i.e.,} \quad \tan^{-1} y = \frac{1}{2}\left(x^2 - 1 \right) + c\, e^{-x^2}$$

160: $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$. (May 97,

: The equation is,

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$$

$$\therefore \quad y^{-1/2}\cdot\frac{dy}{dx} + \frac{x}{1-x^2}y^{1/2} = x$$

Let $y^{1/2} = u$

$$\therefore \quad \frac{1}{2}y^{-1/2}\cdot\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \quad y^{-1/2}\cdot\frac{dy}{dx} = 2\frac{du}{dx}$$

$$\therefore \quad 2\frac{du}{dx} + \frac{x}{1-x^2}u = x$$

$$\therefore \quad \frac{du}{dx} + \frac{1}{2}\frac{x}{1-x^2}\cdot u = \frac{x}{2}$$

ear in u.

$$\therefore \quad \text{I. F.} = e^{\frac{1}{2}\int\frac{x}{1-x^2}dx} = e^{-\frac{1}{4}\int\frac{-2x}{1-x^2}dx}$$

$$= e^{-\frac{1}{4}\log(1-x^2)} = (1-x^2)^{-1/4}$$

neral Solution is,

$$u(1-x^2)^{-1/4} = \int\frac{x}{2}(1-x^2)^{-1/4}\cdot dx + c$$

$$\therefore \quad u(1-x^2)^{-1/4} = -\frac{1}{4}\int(-2x)(1-x^2)^{-1/4}\cdot dx + c$$

$$= -\frac{1}{4}\left[\frac{(1-x^2)^{3/4}}{3/4}\right] + c = -\frac{1}{3}(1-x^2)^{3/4} + c$$

$$\therefore \quad \sqrt{y}(1-x^2)^{-1/4} = -\frac{1}{3}(1-x^2)^{3/4} + c$$

Ex. 18 : Solve : $\frac{dr}{d\theta} = \frac{r \sin\theta - r^2}{\cos\theta}$.

Sol. : We have,

$$\frac{dr}{d\theta} - \frac{r\sin\theta}{\cos\theta} = -\frac{r^2}{\cos\theta}$$

$$\therefore \quad r^{-2}\frac{dr}{d\theta} - r^{-1}\tan\theta = -\sec\theta$$

Let $r^{-1} = u$

$$\therefore \quad -r^{-2}\frac{dr}{d\theta} = \frac{du}{d\theta}$$

$$\therefore \quad -\frac{du}{d\theta} - u\tan\theta = -\sec\theta$$

$$\text{i.e.,} \quad \frac{du}{d\theta} + u\tan\theta = \sec\theta$$

is linear in u

$$\therefore \quad \text{I. F.} = e^{\int \tan\theta \cdot d\theta} = e^{\log\sec\theta} = \sec\theta$$

$\therefore$ General Solution is,

$$u(\sec\theta) = \int \sec\theta \cdot \sec\theta \cdot d\theta = \int \sec^2\theta \cdot d\theta + c$$

$$\therefore \quad u\sec\theta = \tan\theta + c$$

$$\therefore \quad \frac{1}{r}\sec\theta = \tan\theta + c$$ is General

69) $(x^3 y^3 + xy)\frac{dy}{dx} = 1.$ IV

The given equation is,

$$(x^3 y^3 + xy)\frac{dy}{dx} = 1$$

$$\therefore \quad x^3 y^3 + xy = \frac{dx}{dy}$$

$$\therefore \quad \frac{dx}{dy} - xy = x^3 y^3$$

$$\therefore \quad x^{-3}\frac{dx}{dy} - x^{-2} y = y^3$$

Let $x^{-2} = u$

$$\therefore \quad -2x^{-3}\frac{dx}{dy} = \frac{du}{dy}$$

ation becomes

$$-\frac{1}{2}\frac{du}{dy} - uy = y^3$$

$$\therefore \quad \frac{du}{dy} + 2uy = -2y^3$$

r in u.

$$\therefore \quad \text{I. F.} = e^{\int 2y \cdot dy} = e^{y^2}$$

eral Solution is,

$$u\left(e^{y^2}\right) = -2\int e^{y^2} \cdot y^3 \cdot dy + c$$

Let $y^2 = t$

$$\therefore \quad 2y\, dy = dt$$

$$\therefore \quad u \cdot e^{y^2} = -2\int e^t \cdot t \cdot \frac{dt}{2}$$

$$= -\int e^t \cdot t \cdot dt = -\left[t e^t - e^t\right] + c$$

$$\therefore \quad \frac{1}{x^2} e^{y^2} = -y^2 e^{y^2} + e^{y^2} + c$$

i.e., $\frac{1}{x^2} + y^2 - 1 = ce^{-y^2}$ is General Solution

Ex. 63 Solve $\frac{dy}{dx} + \left(\frac{1-2x}{x^2}\right) y = 1.$ **(M.**

: This is a differential equation of the form $\frac{dy}{dx} + Py = Q.$

Now, $\int P\,dx = \int \left(\frac{1-2x}{x^2}\right) dx = \int \frac{dx}{x^2} - 2\int \frac{dx}{x} = -\frac{1}{x} - 2\log x$

$$\therefore \quad e^{\int P\,dx} = e^{-(1/x) - 2\log x} = e^{-1/x} \cdot e^{-2\log x} = e^{-1/x} \cdot \frac{1}{x^2}$$

$\therefore$ The solution is

$$y\,e^{-1/x} \cdot \frac{1}{x^2} = \int e^{-1/x} \cdot \frac{1}{x^2} Q\,dx + c$$

$$= \int e^{-1/x} \cdot \frac{1}{x^2}\,dx + c = e^{-1/x} + c$$

(For integration put $e^{-1/x} = t$)

$\therefore$ The solution is $y\,e^{-1/x} \cdot \frac{1}{x^2} = e^{-1/x} + c$

$$\therefore \quad y = x^2 + c\,e^{1/x} \cdot x^2.$$

b) Solve $(1 + x + xy^2)\,dy + (y + y^3)\,dx = 0.$ **(M.U. 1989, 93, 95)**

have,

$$1 + x(1 + y^2) + y(1 + y^2)\frac{dx}{dy} = 0$$

$$\therefore \quad \frac{dx}{dy} + \frac{x}{y} = -\frac{1}{y(1 + y^2)}$$

is is a linear differential equation of the form $\frac{dx}{dy} + P'x = Q'$.

w, $\int P'dy = \int \frac{dy}{y} = \log y \quad \therefore \quad e^{\int P'dy} = e^{\log y} = y$

The solution is $x \cdot e^{\int P'dy} = \int e^{\int P'dy} \cdot Q'dy + c.$

$$\therefore \quad xy = \int y\left(-\frac{1}{y(1 + y^2)}\right)dy + c$$

$$= -\int \frac{dy}{1 + y^2} = -\tan^{-1} y + c$$

$$\therefore \quad xy + \tan^{-1} y = c.$$

Ex. 3 Solve $(y+1)\,dx + [\,x-(y+2)\,e^{y}\,]\,dy = 0.$ (I

: The equation can be written as

$$\frac{dx}{dy} + \frac{1}{y+1}\cdot x = \frac{y+2}{y+1}\cdot e^{y}$$

It is of the form $\frac{dx}{dy} + P'x = Q'$.

Now, $\int P'dy = \int \frac{1}{y+1}dy = \log(y+1) \quad \therefore\ e^{\int P'dy} = y+1$

$\therefore$ The solution is $x e^{\int P'dy} = \int e^{\int P'dy}\cdot Q'dy + c.$

$$x\cdot(y+1) = \int (y+1)\cdot\frac{(y+2)}{(y+1)}e^{y}dy + c$$

$$= \int (y+2)\cdot e^{y} = (y+2)\,e^{y} - \int e^{y}\cdot dy + c$$

$$= (y+2)\,e^{y} - e^{y} + c = (y+1)\cdot e^{y} + c$$

$$\therefore\ (y+1)(x-e^{y}) = c.$$

Ex. 63 Solve $\frac{dy}{dx} + x^3 \sin^2 y + x \sin 2y = x^3$.

We have $\frac{dy}{dx} + x \sin 2y = x^3(1 - \sin^2 y)$

$$\therefore \quad \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

Dividing by $\cos^2 y$

$$\sec^2 y \frac{dy}{dx} + 2 \tan y \cdot x = x^3$$

Put $\tan y = v$,

$$\sec^2 y \frac{dy}{dx} = \frac{dv}{dx} \qquad \therefore \quad \frac{dv}{dx} + 2x \cdot v = x^3$$

This a linear equation

$$\therefore \quad \int P\,dx = \int 2x\,dx = x^2 \qquad \therefore \quad e^{\int P\,dx} = e^{x^2}$$

$\therefore$ The solution is $v\,e^{x^2} = \int e^{x^2} \cdot x^3\,dx + c.$

To find the integral put $x^2 = t$, $x\,dx = \frac{dt}{2}$.

$$\therefore \quad I = \int e^t \cdot \frac{t}{2}\,dt = \frac{1}{2}[t\,e^t - e^t] = \frac{e^t}{2}(t-1)$$

$$\therefore \quad v\,e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c$$

Resubstituting $v = \tan y$

$$\tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c.$$

Ex. 3 : Solve $y - \cos x \dfrac{dy}{dx} = y^2(1 - \sin x)\cos x.$

Given that $y = 2$ when $x = 0$. (M.U

The given equation can be written as

$$\cos x \frac{dy}{dx} - y = -y^2(1 - \sin x)\cos x$$

Dividing by $-y^2 \cos x$,

$$\therefore \quad -\frac{1}{y^2}\frac{dy}{dx} + \frac{\sec x}{y} = 1 - \sin x$$

Putting $\dfrac{1}{y} = v$ and $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, we get,

$$\frac{dv}{dx} + \sec x \cdot v = 1 - \sin x$$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$.

$$\therefore \quad e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

$\therefore$ The solution is

$$v \cdot (\sec x + \tan x) = \int (\sec x + \tan x)(1 - \sin x)\,dx + c$$

$$= \int \frac{(1 + \sin x)}{\cos x} \cdot (1 - \sin x)\,dx = \int \frac{(1 - \sin^2 x)}{\cos x}\,dx$$

$$= \int \frac{\cos^2 x}{\cos x}\,dx = \int \cos x\,dx = \sin x + c$$

$$\therefore \quad \frac{\tan x + \sec x}{y} = \sin x + c$$

When $x = 0$, $y = 2$. $\quad \therefore \quad 1/2 = c$.

$$\therefore \quad \tan x + \sec x = y\left(\sin x + \frac{1}{2}\right).$$

Ex. 20 Solve $y\dfrac{dx}{dy} = x - yx^2 \sin y$.

: We have $\dfrac{dx}{dy} - \dfrac{x}{y} = -x^2 \sin y$

Dividing by x^2, $\dfrac{1}{x^2}\dfrac{dx}{dy} - \dfrac{1}{x}\cdot\dfrac{1}{y} = -\sin y$

Now, putting $-\dfrac{1}{x} = v$ and $\dfrac{1}{x^2}\dfrac{dx}{dy} = \dfrac{dv}{dy}$,

We get $\dfrac{dv}{dy} + \dfrac{1}{y}v = -\sin y$.

This is a linear differential equation.

$$\therefore \quad e^{\int P\,dy} = e^{\int (1/y)\,dy} = e^{\log y} = y$$

$\therefore$ The solution is

$$vy = \int -\sin y \cdot y\,dy + c$$

$$= [\,y\cos y - \int \cos y \cdot 1 \cdot dy\,] + c$$

$$= y\cos y - \sin y + c$$

$\therefore$ The solution is $-\dfrac{y}{x} = y\cos y - \sin y + c.$

(073)

$$\frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}}$$

is a homogeneous equation

Let $y = vx$

$$\therefore \quad v + x\frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v}$$

$$\therefore \quad \frac{\sec^2 v}{\tan v}dv + \frac{dx}{x} = 0$$

V.S. form, integrate

$$\int \frac{\sec^2 v}{\tan v}dv + \int \frac{dx}{x} = \log c$$

$$\therefore \quad \log\tan v + \log x = \log c$$

$$\therefore \quad x\tan v = c$$

i.e. $x\tan\frac{y}{x} = c$ is General Solution

74 Solve : $\left(x + y\cot\frac{x}{y}\right)dy - y\,dx = 0$

. :

te : Presence of $\frac{x}{y}$ indicates that we use the substitution x = vy.

The equation can be written as

$$x + y\cot\frac{x}{y} - \frac{dx}{dy} = 0$$

Let $x = vy$ $\therefore$ $\frac{dx}{dy} = v + y\frac{dv}{dy}$, equation becomes

$\therefore$ $$vy + y\cot v - y\left(v + y\frac{dv}{dy}\right) = 0$$

$\therefore$ $$y\cot v - y^2\frac{dv}{dy} = 0 \qquad \therefore \cot v - y\frac{dv}{dy} = 0$$

$\therefore$ $$\frac{dy}{y} = \tan v\,dv$$

V.S. form, integration

$$\log y = \log\sec v + \log c$$

$\therefore$ $$y = c\sec v$$

i.e $$y = c\sec\frac{x}{y}$$ is General Solution

75 Solve: $(4x+y)^2 \frac{dx}{dy} = 1$

. : The equation can be written as

$$\frac{dy}{dx} = (4x+y)^2$$

Let $4x + y = u$ $\therefore$ $4 + \frac{dy}{dx} = \frac{du}{dx}$

$\therefore$ The equation is

$$\frac{du}{dx} - 4 = u^2 \qquad \therefore \qquad \frac{du}{u^2+4} = dx$$

V.S. form integrating,

$$\frac{1}{2}\tan^{-1}\frac{u}{2} = x + c$$

i.e. $\tan^{-1}\left(\frac{4x+y}{2}\right) = 2x + c_1$ $\qquad c_1 = 2c$

75) Let $x + y = u$ $\therefore$ $1 + \frac{dy}{dx} = \frac{du}{dx}$

$\therefore$ The equation becomes

$$\cos u \left(\frac{du}{dx} - 1\right) = 1$$

$$\therefore \quad \cos u \cdot \frac{du}{dx} - \cos u = 1$$

$$\therefore \quad \frac{\cos u}{1 + \cos u} du = dx$$

V.S. form, integrating

$$\int \frac{\cos u + 1 - 1}{1 + \cos u} du = \int dx + c$$

$$\therefore \quad \int du - \int \frac{1}{1 + \cos u} du = x + c$$

$$\therefore \quad u - \int \frac{1}{2 \cos^2 \frac{u}{2}} du = x + c$$

i.e. $$u - \frac{1}{2} \int \sec^2 \frac{u}{2} \cdot du = x + c$$

$$\therefore \quad x + y - \frac{1}{2} \cdot 2 \tan \frac{u}{2} = x + c$$

$$\therefore \quad y - \tan\left(\frac{x + y}{2}\right) = c$$

$$\because \quad u = x + y$$

is G.S.

17 Solve: $\left(\frac{y}{x}\cos\frac{y}{x}\right)dx - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)dy = 0$

I. :

Let $\frac{y}{x} = v$

$\therefore\ y = vx \qquad \therefore\ \frac{dy}{dx} = v + x\frac{dv}{dx}$

The given equation is

$$\frac{y}{x}\cos\frac{y}{x} - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)\frac{dy}{dx} = 0$$

$$v\cos v - \left(\frac{1}{v}\sin v + \cos v\right)\left(v + x\frac{dv}{dx}\right) = 0$$

After simplification, we get

$$v\sin v = -x(\sin v + v\cos v)\frac{dv}{dx}$$

$$\frac{dx}{x} + \frac{\sin v + v\cos v}{v\sin v}dv = 0$$

V.S. form, integrating, we get

$$\int\frac{dx}{x} + \int\frac{\sin v + v\cos v}{v\sin v}dv = c$$

$$\log x + \log(v\sin v) = c$$

$$xv\sin v = e^c = c_1$$

e. $$y\sin\frac{y}{x} = c_1 \text{ is General solution}$$

79 Solve: $xdy - ydx = x\sqrt{x^2+y^2}\cdot dx$ (II/01)

I. :

Let $\frac{y}{x} = u$ i.e. $y = ux$

$$\therefore \quad \frac{dy}{dx} = u + x\frac{du}{dx}$$

The equation is

$$x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$$

Substituting, we have

$$x\left(u + x\frac{du}{dx}\right) - ux = x\sqrt{x^2+u^2x^2}$$

$$\therefore \quad u + x\frac{du}{dx} - u = x\sqrt{1+u^2}$$

$$\therefore \quad \frac{du}{\sqrt{u^2+1}} = dx$$

V.S. form, integrating, we get

$$\int\frac{du}{\sqrt{u^2+1}} = \int dx + c_1$$

$$\log\left(u+\sqrt{u^2+1}\right) = x + c_1$$

i.e.

$$\log\left(\frac{y}{x} + \frac{\sqrt{y^2+x^2}}{x}\right) = x + c_1$$

$$\therefore \quad \frac{y+\sqrt{y^2+x^2}}{x} = c\,e^x \; (e^{c_1} = c)$$

$$\therefore \quad y+\sqrt{y^2+x^2} = cx\,e^x$$ is General solution

79 : Solve : $x\,dx = y\,(x^2 + y^2 - 1)\,dy$

Sol. : The equation is

$$x\,dx + y\,dy = y\,(x^2 + y^2)\,dy$$

Let $x^2 + y^2 = u$

$\therefore \quad 2\,(xdx + ydy) = du$

$\therefore$ Equation becomes,

$$\frac{du}{2} = y\,u\,dy$$

$$\therefore \quad \frac{du}{u} = 2y\,dy$$

V.S. form, integrating, we get

$$\log u = y^2 + c$$

i.e. $\log\,(x^2 + y^2) = y^2 + c$

e(:80) $x + y\frac{dy}{dx} = \left(x\frac{dy}{dx} - y\right)\sqrt{\frac{a^2 - x^2 - y^2}{(x^2 + y^2)}}$

:

Let $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $x\,dx + y\,dy = r\,dr$

and $x\,dy - y\,dx = r^2\,d\theta$

The given equation is

$$x\,dx + y\,dy = (x\,dy - y\,dx)\sqrt{\frac{a^2 - (x^2 - y^2)}{(x^2 + y^2)}}$$

From (1) and (2), $r\,dr = (r^2\,d\theta)\frac{\sqrt{a^2 - r^2}}{r}$

$\therefore$ $$\frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

V.S. form, integrating,

$$\sin^{-1}\frac{r}{a} = \theta + c$$

r $\sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{a}\right) = \tan^{-1}\frac{y}{x} + c$ is General solution

.

(159)

u = x + y

$$\therefore \quad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

∴ equation becomes

$$\left(\frac{du}{dx} - 1\right) + xu = x^3 u^3 - 1$$

$$\therefore \quad \frac{du}{dx} + xu = x^3 u^3$$

is Bernoulli's equation

Dividing by u^3, we get

$$u^{-3}\frac{du}{dx} + x\,u^{-2} = x^3$$

$$\text{Let} \quad u^{-2} = v$$

$$\therefore \quad -2\,u^{-3}\frac{du}{dx} = \frac{dv}{dx}$$

$$\therefore \quad -\frac{1}{2}\frac{dv}{dx} + xv = x^3$$

$$\therefore \quad \frac{dv}{dx} - 2xv = -2\,x^3 \quad \text{is linear in v}$$

$$\text{I.F.} = e^{-2\int x\,dx} = e^{-x^2}$$

∴ General solution is,

$$v\,e^{-x^2} = -2\int x^3 e^{-x^2}\cdot dx + c$$

$$\text{Let } x^2 = t$$

$$\therefore \quad 2xdx = dt$$

$$v\,e^{-x^2} = -\int t\,e^{-t}\cdot dt + c = -[\,t(-e^{-t}) - (1)(e^{-t}$$

$$\therefore \quad v\,e^{-x^2} = t\,e^{-t} + e^{-t} + c$$

$$\therefore \quad \frac{1}{(x+y)^2}e^{-x^2} = e^{-x^2}(x^2+1) + c$$

$$\text{or} \quad \frac{1}{(x+y)^2} = (x^2+1) + c\,e^{x^2} \quad \text{is General solu}$$

82 Solve : $(x^2+2)\,y^3\,dx + (x^3+y^3)\,(y\,dx - x\,dy) = 0$ VII

∴ The presence of (y dx− x dy) indicating that we substitute.

$$\frac{y}{x} = v$$

$$\therefore \quad dv = \frac{x\,dy - y\,dx}{x^2}$$

∴ Equation becomes,

$$(x^2+2)\,v^3\,x^3\,dx + (x^3+v^3x^3)\,(-x^2\,dv) = 0$$

$$\therefore \quad (x^2+2)\,v^3\,dx - x^2\,(1+v^3)\,dv = 0$$

$$\therefore \quad \frac{x^2+2}{x^2}\,dx - \frac{1+v^3}{v^3}\,dv = 0$$

V.S. form, integrating,

$$\int \frac{x^2+2}{x^2}\,dx - \int \frac{v^3+1}{v^3}\,dv = c$$

i.e. $$x - \frac{2}{x} - v + \frac{1}{2v^2} = c$$

i.e. $$x - \frac{2}{x} - \frac{y}{x} + \frac{x^2}{2y^2} = c$$ is General solution

'03 Solve : $4x^2 y \frac{dy}{dx} = 3x(3y^2 + 2) + (3y^2 + 2)^3$

∴ The factor $(3y^2 + 2)$ appears predominant

$$\therefore \quad \text{Let} \quad 3y^2 + 2 = u$$

$$\therefore \quad 6y\frac{dy}{dx} = \frac{du}{dx}$$

$$4x^2\left(\frac{1}{6}\frac{du}{dx}\right) = 3\,xu + u^3$$

$$\therefore \quad \frac{2x^2}{3}\frac{du}{dx} - 3\,xu = u^3$$

is Bernoulli's equation

Dividing throughout by $\frac{2x^2}{3}u^3$, we get

$$u^{-3}\frac{du}{dx} - 3x\left(\frac{3}{2x^2}\right)u^{-2} = \frac{3}{2\,x^2}$$

$$\text{Let} \quad u^{-2} = v$$

$$\therefore \quad -2\,u^{-3}\frac{du}{dx} = \frac{dv}{dx}$$

$$\therefore \quad -\frac{1}{2}\frac{dv}{dx} - \frac{0}{2x}v = \frac{3}{2x^2}$$

$$\text{i.e.} \quad \frac{dv}{dx} + \frac{9}{x}v = -\frac{3}{x^2} \quad \text{is linear in v}$$

$$\text{I.F.} = e^{\int \frac{9}{x}dx} = e^{9\log x} = x^9$$

∴ General solution is

Ex. 98: $[y\sin(xy) + xy^2\cos(xy)]\,dx + [x\sin(xy) + x^2 y\cos(xy)\,dy] = 0.$

Sol. : We write the equation as

$$[\sin(xy) + xy\cos(xy)]\,y\,dx + [\sin(xy) + xy\cos(xy)]\,x\,dx = 0$$

$$\therefore \quad [\sin xy + xy\cos(xy)](y\,dx + x\,dy) = 0$$

$$\text{Let } u = xy$$

$$\therefore \quad du = x\,dy + y\,dx;$$

$$\text{and} \quad [\sin u + u\cos u]\,du = 0$$

Integrating; we get

$$\int \sin u \cdot du + \int u\cos u \cdot du = c$$

$$\text{i.e,} \quad -\cos u + [u\sin u - (1)(-\cos u)] = c$$

$$\therefore \quad u\sin u = c$$

$$\text{i.e.} \quad xy\sin(xy) = c \text{ is General Solution.}$$